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The possibility of thermal solitons

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INTRODUCTION

It is well known that the heat conduction in a material with thermal memory is described by the general hyperbolic heat equation of the type (see e.g. [1] and the references therein)

$$\tau\frac{\partial^2 T}{\partial t^2}+\frac{\partial T}{\partial t}-\alpha\Delta T = q(t,x,y,z,T) \quad (1)$$

where τ is the thermal relaxation time, α is the thermal diffusivity, Δ is Laplace operator and $q(t,x,y,z,T)$ is a source term. This equation forms the basis for the experimental determination of the coefficient of the thermal diffusivity α and the relaxation time τ of the material with the memory by means of the so-called flash method [2]. For this purpose the slab of the investigated material of the thickness L (L small) is considered which is described by one-dimensional version of equation (1)

$$\tau\frac{\partial^2 T}{\partial t^2}+\frac{\partial T}{\partial t}-\alpha\frac{\partial^2 T}{\partial x^2} = q(t,x,T) \quad (2)$$

initially at the equilibrium temperature $T(x,0) = 0$. At the time $t = 0$ the external surface at the point $x = 0$ is suddenly exposed to a certain time-dependent heat pulse with a prescribed time function. The heat pulse travels through the slab and is determined by the solution of the differential equation (1) under the given boundary conditions [3]. The experimental value of the thermal diffusivity α and the relaxation time τ are then deduced from the spatial and temporal response field of the travelling pulse in the investigated material [4]. When determining the thermal parameters of a thermal medium we do not know *a priori* whether it is a standard material described by the parabolic heat equation

$$\frac{\partial T}{\partial t}-\alpha\frac{\partial^2 T}{\partial x^2} = q(x,t,T) \quad (3)$$

where T is a temperature at the space-time point (x, t) or a material with thermal memory described by the hyperbolic heat-conduction equation (1).

There exist a great number of papers which deal with the solution of equation (1), especially those which take into account different boundary conditions occurring in an experimental arrangement. They develop mathematical methods how to distinguish two types of materials from the propagating response and then determine their thermal parameters (see e.g. [5] and [6]). The overwhelming majority of the authors consider generally the source term in equation (1) as a function of time only (i.e. describing a kind of a heat pulse).

The aim of this paper is to show theoretically that in media with memory described by equation (2) with a special choice of the source function $q(T)$ there can be generated thermal solitary waves or solitons under certain boundary conditions. In the theory of wave propagation it is well-known that besides the travelling waves there can exist in certain media and at certain physical conditions also so-called solitary waves or solitons [7]. They represent stable localized wave packets which do not disperse and are special solutions of certain nonlinear wave equations. For example, there exists a class of physical problems which are described by the sine–Gordon model [8, 9]

$$\phi_{tt}-\phi_{xx}+\sin\phi = 0 \quad (4)$$

or by the ϕ^4 model [8]

$$\phi_{tt}-\phi_{xx}-\phi+\phi^3 = 0. \quad (5)$$

The non-linear equations (4) and (5) possess travelling soliton (4) and solitary wave (5) solutions, $\phi[(x-vt)/(1-v^2)^{1/2}]$, with arbitrary velocity v, which represent 'quasiparticles' of the respective non-linear field theories [10]. At certain conditions also additional perturbations do not destroy their stability and the travelling nature of the solitons: if on the RHS of equations (4) and (5) appears the function $F(x,t) = f-\Gamma\phi_t$, where f and Γ are constant external driving force (f small, $f \ll 1$) and a friction coefficient, respectively, then the soliton solution $\phi[(x-vt)/(1-v^2)^{1/2}]$ can be saved if the friction losses of the energy are compensated by the energy supply due to the external driving field f, [11, 12]

$$\Gamma\int_{-\infty}^{+\infty}\mathrm{d}x\,\phi_t^2+f\int_{-\infty}^{+\infty}\mathrm{d}x\,\phi_t = 0. \quad (6)$$

When using the respective soliton solution for ϕ in equation (6) we get an equation for the velocity v of the soliton; evidently it is no more arbitrary, but it becomes a function of both perturbation parameters, $v = v(f/\Gamma)$ [12, 13].

However, for the case with damping and driving forces of the type (6) the usual soliton solution in the form $\phi[(x-vt)/(1-v^2)^{1/2}]$ must be generalized as $\phi[(x-x_0(t))/(1-\dot{x}_0^2(t))^{1/2}]$. Namely, the damping and driving forces interplay so that during some transient time they reach asymptotically an equilibrium with the constant value of the velocity, where $x_0(t) = vt$ [14, 15]. We will not consider

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NOMENCLATURE

f	constant driving force
L	thickness of the sample
q	source term
T	temperature
v	travelling velocity of the soliton.

Greek symbols

α	thermal diffusivity
Δ	Laplace operator
Γ	friction coefficient
τ	thermal relaxation time
ϕ	amplitude of the soliton.

these transient effects in what follows so that the results will be valid in the asymptotic region.

Equation (2) in one space dimension can be rewritten into the normal form usual in soliton physics

$$\frac{1}{v_0^2}\frac{\partial^2 T}{\partial t^2}-\frac{\partial^2 T}{\partial x^2}+\Gamma\frac{\partial T}{\partial t}=\frac{1}{\alpha}q \tag{7}$$

where $v_0^2 = \alpha\tau^{-1}$ and $\Gamma = \alpha^{-1}$.

With the variable $\xi = (x-vt)/(1-v^2/v_0^2)^{1/2}$ equation (7) becomes

$$\frac{d^2T}{d\xi^2}+\frac{v}{\alpha}(1-v^2/v_0^2)^{-1/2}\frac{dT}{d\xi}+\frac{1}{\alpha}q=0. \tag{8}$$

If we take

$$\frac{q}{\alpha}\equiv -f+T-T^3 \tag{9a}$$

or

$$\frac{q}{\alpha}\equiv -f+\sin T \tag{9b}$$

then equation (8) has the following solutions:

(i) With the choice (9a) the solitary wave is a travelling domain wall given by [12, 14]

$$T(\xi) = T_0+T_1\tanh\gamma\xi \tag{10}$$

where

$$T_0 = \nu_3/2,\quad T_1 = \pm(1-\tfrac{3}{4}\nu_3^2)^{1/2}\quad \gamma=(1/\sqrt{2})T_1. \tag{11}$$

Here, ν_3 is a solution to equation

$$\nu^3-\nu+f=0,\quad \nu_1<\nu_3<\nu_2.$$

If f is small, then $\nu_3 \sim f$. The condition for the solution (10) to exist is the identity

$$\frac{v}{\alpha}\left(1-\frac{v^2}{v_0^2}\right)^{-1/2} = \pm\frac{3}{\sqrt{2}}\nu_3. \tag{12}$$

From (12) and for small f the travelling velocity v of the solitary wave yields

$$v=\frac{3}{\sqrt{2}}\alpha\nu_3(1+\tfrac{9}{2}\alpha\tau\nu_3^2)^{-1/2}\approx\frac{3}{\sqrt{2}}\alpha f(1+\tfrac{9}{2}\alpha\tau f^2)^{-1/2}. \tag{13}$$

Hence, for small f and for $\frac{9}{2}\alpha\tau f^2 \ll 1$, the velocity v is very weakly dependent on the relaxation time τ. It is determined only by α and by the parameter f of the source function, $v \sim (3/\sqrt{2})\alpha f$.

(ii) For the periodic source function (9b) the soliton solution of equation (8) is of the form (see e.g. [15, 16]),

$$T(x,t) = 4\sigma\arctan\exp\left[(x-vt)\Big/\left(1-\frac{v^2}{v_0^2}\right)^{1/2}\right]+\phi_s \tag{14}$$

where

$$\phi_s = \arcsin f\quad |f|<1\quad \sigma=\pm 1\quad v=\left[1+\left(\frac{4}{\pi\alpha f}\right)^2\right]^{-1/2}. \tag{15}$$

For $(\pi f\alpha/4)^2 \ll 1$, we have

$$v\approx\frac{\pi f\alpha}{4}. \tag{16}$$

Let us note that for small f the dependence $v = v(f\alpha)$ is similar in both cases (i) and (ii) up to a numerical factor; this is evident from the comparison of equations (13) and (15) or (16).

From what has been said so far it follows that under certain physical conditions in a medium with memory the thermal soliton can be generated and its form and propagation velocity are functionally linked with the medium thermal parameters. Experimental verification of this theoretical conclusion requests, however, a sophisticated experimental arrangement. Especially, the exact form of the non-linear source term of equation (6) is difficult to realize experimentally. On the other hand, the experimental identification of thermal solitons would enrich the thermal physics and could have certain practical impact also for the determination of thermal material parameters. The detailed discussion of the possibility for experimental generation of thermal solitons exceeds the scope of this note, therefore it will be a subject of a subsequent paper. The aim of this short note was only to point out the theoretical possibility of the existence of thermal solitons in media with thermal memory.

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